CERTAIN ALGEBRAIC SOLUTIONS OF THE RICCATI EQUATION AND ELLIPTIC FUNCTIONS

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Abstract. We study the irreducible and algebraic equation

$$x^n + 4a_1x^{n-1} + \cdots + a_n = 0,\ n \geq 4 \tag{1}$$

on a differential field $(\mathbb{F} = \mathcal{C}(x), \delta =')$ of characteristic zero with algebraically closed field of constants $\mathcal{C}$. We assume that a root of the latter equation is solution of a Riccati differential equation

$$u' = B_0 + B_1u + B_2u^2$$

where B_0, B_1, B_2 are in $\mathbb{F}$.

We obtain a parameterization of the polynomials in terms of a variable $T \in \mathbb{F}$. All the possible solutions are of the form

$$F(x, T) = 0$$

with numerical coefficients. We also give the possible algebraic Galois groups in each case and build explicitly the concerned Riccati equations. We then give an example of a degree 3 irreducible polynomial equation satisfied by certain weight 2 modular forms (for the subgroup $\Gamma(2) = \{M \in SL(2,\mathbb{Z}), M = \mathrm{Id} \mod 2\}$ of $SL(2,\mathbb{Z})$), all whose solutions satisfy a same Riccati equation on the differential field $\left(\mathbb{C}(E_2, E_4, E_6), \dfrac{d}{d\tau}\right)$ with E_i the Eisenstein series of weight i respectively. These latter solutions are related to a Darboux-Halphen system. We then show that in the generic situation, corresponding to when the three solutions of the Darboux-Halphen system are all distinct, these latter satisfy an irreducible degree three polynomial on the differential field $\left(\mathbb{C}(\gamma, \gamma', \gamma''), \dfrac{d}{d\tau}\right)$, with γ the generic solution of the Chazy differential equation $\gamma''' = 6\gamma''\gamma - 9\gamma'^2$. Finally our interest is the following problem: for which potential q does the Riccati equation $\dfrac{du}{dz} + u^2 = q$ admit algebraic solutions over the differential field $\mathbb{C}(\wp, \wp')$. In this problem, the approach will be through Darboux polynomials.

Contents

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1. Introduction

Let $(\mathbb{F} = \mathcal{C}(x), \delta =')$ be a differential field of characteristic zero with algebraically closed field of constants $\mathcal{C}$. We proceed in this paper to the study of some particular properties of the Riccati equation which is a differential equation of the form signaled above. We consider an algebraic irreducible equation $h_n(x)$ of degree $n \geq 4$ (1)

$$h_n(x) = x^n + a_1(t)x^{n-1} + \cdots + a_n(t) = 0, \quad n \geq 4 \tag{2}$$

where the a_i reside in the field $\mathbb{F}$. Suppose a root of h_n is solution of a Riccati equation

$$u'(t) = B_0 + B_1 u + B_2 u^2, \tag{3}$$

where B_0, B_1, B_2 are in $\mathbb{F}$. As we are in characteristic zero, every irreducible polynomial is separable and hence has got a transitive Galois group H. So if one root of equation (2) satisfies the equation (3), all its roots do. Therefore our hypothesis is identical to giving oneself an irreducible separable $h_n(x)$ all whose roots satisfy a same Riccati equation of the form (3).

The cross-ratio of any four solutions x_i ([23]) will be a constant and we will call such equations, anharmonic equations. This fundamental property leads to profound implications; it is indeed this constance which will enable us to effectively parameterize the coefficients of the anharmonic polynomials. In the following we will not distinguish between the h_n (2) which differ only from a substitution

$$x \to \frac{Ax + B}{Cx + D} \tag{4}$$

with $AC - BD \neq 0$ and A, B, C, $D \in \mathbb{F}$.

This identification makes sense because such a transformation changes a solution of a Riccati equation to another solution of a Riccati differential equation and preserves the cross-ratio of any four elements in some field extension of $\mathbb{F}$ and does not change the irreducible nature of h_n.

We will take advantage of similar substitutions in order to assume that the sum a_1 of the roots of f is zero or to multiply x by an appropriate element of $\mathbb{F}$. The goal being the simplification of the expressions by associating to a given anharmonic equation an equivalent simpler one.

This is the main result that we obtain

Theorem 1. *Every h_n is of the form*

$$F(x, T) = 0,$$

where the polynomial in two variables F has constant coefficients (elements of $\mathcal{C}$) and $T \in \mathbb{F}$. Its algebraic Galois group is a finite subgroup of $PGL(2, \mathcal{C})$.

The interest in the Riccati equation has always been huge; it is the only first order differential equation (solved in the first derivative) up to fractional linear transformations of the dependent variable and holomorphic transformation of the independent variable, possessing the Painlevé property. To be more precise, one knows from the Cauchy theorem of existence and unicity of a local solution of a differential equation, that the general solution of a first order differential equation is of the form

$$f(z, C_0), \quad \text{with } C_0 \text{ an arbitrary constant of } \mathbf{C}.$$

But in general the solution $y(z, C_0)$ exhibits singular points, meaning points where $y(z, C_0)$ is not analytical. These singularities are of different types: poles, or possibly less friendly ones called branch points and essential singularities.

In most cases the location of these singularities depends on the constant C_0. The Riccati equation is the only first order equation whose only movable singularities are poles (Painlevé property). One may consult [14] for the latter subject.

Another aspect of the theory of Riccati equations is the linearization property. In fact the substitution

$$u = -\frac{z'}{B_2 z},$$

changes the Riccati equation into a second order linear one. Therefore the particular Riccati equation

$$\frac{du}{dt} = u^2 + Q,$$

is related in this way to the one dimensional Schrödinger equation.

$$\frac{d^2 z}{dt^2} = -Qz.$$

Nowadays the Riccati equation is most notably used in the field of linear differential Galois theory where it is used in order to characterize existence of Liouvillian solutions for the second order linear differential equation [29, 30, 31]. This paper is structured as follows: in section 2 we give a review of differential Galois theory, prove that the Galois groups of anharmonics are subgroups of $PGL(2, \mathcal{C})$, give the explicit form of the Riccati equations and end with the main theorem. Then we give the possible degrees of anharmonics and example in subsection 2.1. Then we study the irreducible degree 3 polynomials coming from modular forms and the Darboux-Halphen system. Finally in the last section 4 we are interested in the following problem: for which potentials q does the Riccati equation $\frac{dy}{dx} + y^2 = q$ admit algebraic solutions over the differential field $\mathbb{C}(\wp, \wp')$.

2. Algebraic solutions of the Riccati equation

We first remind some facts about the Galois theory of second order linear differential equations. One may consult [27] for a more complete exposition about the subject.

A differential field $(\mathbb{F}, \delta)$ is a field together with a derivation δ on $\mathbb{F}$. One says that a differential field $(\mathfrak{F}, \Delta)$ is a differential field extension of $(\mathbb{F}, \delta)$ if $\mathfrak{F}$ is a field extension of $\mathbb{F}$ and its derivation Δ extends δ.

In the following y' and y'' will stand for $\delta(y)$ and $\delta^2(y)$. Let an ordinary homogeneous second order linear differential equation

$$L(y) = y'' + b_1 y' + b_0 y = 0, \quad a_i \in \mathbb{F}.$$

Let η, ζ a fundamental set of solutions of L (two independent solutions generating its two dimensional vector space of solutions V over $\mathcal{C}$). Form the differential extension field

$$\mathfrak{F} = \mathbb{F} < \eta, \zeta >= \mathbb{F}(\eta, \eta', \zeta, \zeta').$$

The former field extension $\mathfrak{F}$ is called a Picard-Vessiot extension if in addition $\mathfrak{F}$ and $\mathbb{F}$ have the same field of constants. The differential Galois group $\mathfrak{G}(L)$ of $\mathfrak{F}$ over $\mathbb{F}$ is the group of differential automorphisms of $\mathfrak{F}$ that leave $\mathbb{F}$ invariant. (An automorphism σ is differential if $\sigma(a') = (\sigma a)'$, $a \in \mathfrak{F}$).

The previous choice of a basis of solution induces a faithful representation of $\mathfrak{G}(L)$ as a subgroup of $GL(2,\mathcal{C})$, defined in the following way: for $\sigma \in \mathfrak{G}(L)$ one has

$$\begin{cases} \sigma(\eta) = a_\sigma \eta + b_\sigma \zeta \\ \sigma(\zeta) = c_\sigma \eta + d_\sigma \zeta. \end{cases}$$

A different different choice of basis (η_1, ζ_1) leads to equivalent representations as there exists $M \in GL(2,\mathcal{C})$: $(\eta_1,\zeta_1) = M(\eta,\zeta)$. We identify these equivalent representations. Also one can show, (see [27, p.19]), that the differential Galois group considered as a subgroup of $GL(2,\mathcal{C})$ is an algebraic subgroup.

We have the following Galois correspondence [27, p.25]

Proposition 2. *Let $L(y) = y'' + a_1 y' + a_2 y = 0$ be a second order equation over $\mathbb{F}$ with Picard-Vessiot field $\mathfrak{F}$ and $\mathfrak{G}(L) := Gal(\mathfrak{F}/\mathbb{F})$ its differential Galois group. Consider the two sets*

$$\mathfrak{S} := \textit{the closed algebraic subgroups of } \mathfrak{G}(L) \textit{ (in the Zariski Topology) and}$$

$$\mathfrak{L} := \textit{the differential subfields } M \textit{ of } \mathfrak{F}, \textit{ containing } \mathbb{F}.$$

Define $\alpha : \mathfrak{S} \to \mathfrak{F}$ by $\alpha(H) = \mathfrak{F}^H$, the subfield of H-invariant elements of $\mathfrak{F}$. Let $\beta : \mathfrak{L} \to \mathfrak{S}$ defined by $\beta(M) = Gal(\mathfrak{F}/M)$, be the group of M-linear differential automorphisms. Then

(1) *The maps α and β are mutual inverses.*
(2) *The subgroup $H \in \mathfrak{S}$ is normal if and only if $M = \mathfrak{F}^H$ is, as a set, invariant under G. If $H \in \mathfrak{S}$ is normal then the canonical map $\mathfrak{G}(L) \to Gal(M/\mathbb{F})$ is surjective and has kernel H. Moreover, M is a Picard-Vessiot field for some linear differential equation over $\mathbb{F}$.*
(3) *Let $\mathfrak{G}(L)^0$ denote the identity component of $\mathfrak{G}(L)$. Then $\mathfrak{F}^{\mathfrak{G}(L)^0} \supset \mathbb{F}$ is a finite Galois extension with Galois group $\mathfrak{G}(L)/\mathfrak{G}(L)^0$ and is the algebraic closure of $\mathbb{F}$ in $\mathfrak{G}(L)$.*

Now the mapping $y \mapsto \dfrac{y'}{y}$ defines a surjection between the set of non trivial solutions of $L(y) = y'' + a_1 y' + a_2 y = 0$ and the solutions of the Riccati equation $u' + b_1 u^2 + b_2 u + b_3 = 0$, with values b_i function of a_i and their derivatives. In the sequel we assume we deal with this form of the Riccati equation, which we suppose to have solutions which are algebraic of degree≥ 4. We remind that $h_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n$, $a_i \in \mathbb{F}$ denotes its minimal polynomial. Let x_i, $i = 1, \cdots, n$ be its roots in a decomposition field. There exists $\dfrac{y_i'}{y_i}$, $i = 1, \cdots, n$: $x_i = \dfrac{y_i'}{y_i}$. Using a fundamental basis of solution (η,ζ) of $y'' + a_1 y' + a_2 y$, one sees that the u_i all belong to the Picard-Vessiot extension $\mathfrak{F} = \mathbb{F} < \eta, \zeta >= \mathbb{F}(\eta, \eta', \zeta, \zeta')$. But the field extension $\mathbb{F}[x_i,\, i = 1, \cdots, n]$ of $\mathbb{F}$ is a differential subfield of the Picard-Vessiot extension as $x_i' + b_1 x_i^2 + b_2 x_i + b_3 = 0$. Thus it corresponds by the Galois correspondence to a finite closed algebraic subgroup H of the differential Galois group $\mathfrak{G}(L) \subset GL(2,\mathcal{C})$. These differential automorphism of $\mathbb{F}[x_i,\, i = 1, \cdots, n]$ are in the algebraic Galois group of h_n. Conversely let σ belong to the algebraic Galois group of h_n; by hypothesis it is transitive and permutes the roots x_i of h_n. If $\sigma(x_i) = x_j$ for $i \neq j$, the case $i = j$ being clear, one has

$$\sigma(x_i)' = x_j' = -(b_1 x_j^2 + b_2 x_j + b_3)$$

and

$$\sigma(x_i') = \sigma(-(b_1 x_i^2 + b_2 x_i + b_3)) = -b_1 \sigma(x_i)^2 - b_2 \sigma(x_i) - b_3 = -b_1 x_j^2 - b_2 x_j - b_3;$$

therefore extending to the algebra $\mathbb{F}[x_i,\, i = 1, \cdots, n]$, one sees that any element of the algebraic Galois group is also differential and fixes $\mathbb{F}$, ie belongs to H. Now one remarks that any element of $\mathfrak{G}(L) \bigcap \mathcal{C}^\star$: the subgroup of those $\sigma \in \mathfrak{G}(L)$ that act on the solution space V as scalar multiplication, has a trivial action on any $\dfrac{y'}{y}$, for y any non-trivial solution of $L(y) = y'' + a_1 y' + a_2 y = 0$.

Thus one has, passing to quotient space the following lemma (as $\mathcal{C}$ is algebraically closed one has $PSL(2,\mathcal{C}) = PGL(2,\mathcal{C})$).

Proposition 3. *The Galois group of any h_n is a finite algebraic subgroup of $PSL(2,\mathcal{C}) = PGL(2,\mathcal{C})$, with algebraic meaning that it is the image under the quotient map of a finite algebraic subgroup of $SL(2,\mathcal{C})$.*

The list of the finite algebraic subgroups of $SL(2,\mathcal{C})$ is furnished in [17, 18]. We have

Theorem 4 ([17, 18])**.** *Let G be a finite subgroup of $SL(2,\mathcal{C})$. Then either*

- *G is conjugate to a subgroup of the group*

$$D^{\dagger} = D \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .D$$

  with D the diagonal group, or
- *the order of G is 24 (the tetrahedral case), or*
- *the order of G is 48 (the octahedral case), or*
- *the order of G is 120 (the icosahedral case).*

In the last three cases G contains the scalar matrix -1.

These groups admit well-known representations as fractional linear transformations (see [17]). The tetrahedral group A_4 of $PSL(2,\mathcal{C})$ is isomorphic to the group of linear fractional transformations generated (under composition) by the elements

$$\Theta_2 : z \to -z; \quad \epsilon_0 : z \to \frac{1}{z}; \quad \theta_1 : z, \frac{1-i}{1+i}\frac{z+1}{z-1}$$

the dihedral group $\mathfrak{D}_m$ is isomorphic with the subgroup of linear fractional transformations generated by

$$\Theta_m : z \to \xi z; \quad \epsilon_0 : z \to \frac{1}{z}, \text{ where } \xi^m = 1.$$

The octahedral group is on his side generated by the two linear fractional transformations

$$\theta_1 : z \to \frac{1-i}{1+i}\frac{z+1}{z-1}, \quad \theta_2 : z \to \frac{1-i}{1+i}z;$$

the icosahedral comes from the linear fractional transformations

$$\Theta_5 : z \to \xi z \quad \epsilon_0 : z \to \frac{1}{z}, \quad \epsilon_1 : z \to \frac{\alpha z + \beta}{\beta z - \alpha}, \text{ with } \xi^5 = 1,\ \alpha = \frac{\xi^4 - \xi}{\sqrt{5}} \text{ and } \beta = \frac{\xi^2 - \xi^3}{\sqrt{5}}.$$

Finally the cyclic subgroups of order n are generated by

$$\Theta_n : z \to \xi z,\ \xi^n = 1.$$

To each of the previous groups of fractional linear transformations is associated an absolute invariant or "automorphic form".

For the tetrahedral case, an absolute invariant is given by the formula

$$\Psi_{tetr}(z) = \frac{\psi(z)}{\phi(z)} = \left(\frac{z^4 + 1 + 2i\sqrt{3}z^2}{z^4 + 1 - 2i\sqrt{3}z^2}\right)^3.$$

For the dihedral cases one can consider the map

$$\Psi_{dih}(z) = \frac{\psi(z)}{\phi(z)} = \frac{1}{z^m} + z^m.$$

Absolute invariants for the octahedral and icosahedral groups read respectively

$$\Psi_{oct}(z) = \frac{\psi(z)}{\phi(z)} = \frac{(1+14z^4+z^8)^3}{108z^4(1-z^4)^4}$$

and

$$\Psi_{ico}(z) = \frac{\psi(z)}{\phi(z)} = \frac{(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3}{1728z^5(z^{10}+11z^5-1)^5}.$$

The cyclic group admits the invariant

$$\Psi_{cyc}(z) = \frac{\psi(z)}{\phi(z)} = z^n$$

The properties of the absolute invariants Ψ are the following:

- They are left invariant under the transformation of the concerned groups, with the action given by
$$(h_f, \Psi) \to h_f.\Psi : z \to \Psi(h_f.z).$$
- Every other absolute invariant for the group is a rational function with coefficients in $\mathcal{C}$ of Ψ.

Before continuing we state the constance of the cross-ratio of four solutions of the Riccati equation

Lemma 5. *Let $(\mathbb{F}, \delta =')$ a differential field and $u' + b_1u^2 + b_2u + b_3 = 0$ a Riccati differential equation with coefficients in $\mathbb{F}$. Let u_1, u_2, u_3, u_4 four arbitrary distinct solutions of the Riccati equation. Form the cross-ratio*

$$\frac{(u_1-u_3)(u_1-u_4)}{(u_2-u_3)(u_2-u_4)}$$

and take the derivative with respect to δ; then it vanishes identically.

These absolute invariants will be useful in order to express the roots of the anharmonics. We will then use the constance of the cross-ratio to determine a form for the general solution of the Riccati equation and finally build the Riccati equation into consideration explicitly.

For each anharmonic h_n with algebraic Galois group $H \subset PSL(2,\mathcal{C})$ (as represented above), consider one of its roots, x_i, which is therefore solution of the associated Riccati equation $u' + b_1u^2 + b_2u + b_3 = 0$. Let $H \subset PSL(2,\mathcal{C})$ the Galois group of h_n and $\mathfrak{t}$ a variable then

$$\Psi(\mathfrak{t})$$

is left invariant by H; therefore belongs to $\mathbb{F}$. Consider $T \in \mathbb{F} \setminus \mathcal{C}$.

Let H_{x_i} the stabilizer of a root of h_n. As the Galois group of h_n is transitive, one has if N denotes its order, $N = np$, where p is the cardinal of the stabilizer. Consider the action of H_{x_i} on the set of roots x_i of h_n. Two cases can happen. H_{x_i} does not fix any other root $x_j \neq x_i$; then its action on a root $x_j \neq x_i$ gives rise to an orbit $H_{x_i}x_j$ of length p. By the disjointness of the different orbits and the fact that the total sum of their cardinals is $n-1$, one has: $p|(n-1)$. In the same way one sees that $p|(n-2)$ if it happens that H_{x_i} fixes another root $x_j \neq x_i$.

No element of H different from the identity element can fix more than two roots by constance of the cross-ratio. Regarding now H as an homography group, we denote by η_i, $i = 1, \cdots, n$ a fixed-point of the corresponding homography subgroup of H_{x_i}, S_i. We first have the following lemma which shows existence of fixed points for finite homography subgroups.

Lemma 6. *Every finite homography subgroup admits a fixed point.*

Proof. Let the equation

$$\frac{ax+b}{cx+d} = x,\ ad-bc \neq 0$$

hold; then one has

$$cx^2 + (d-a)x - b = 0.$$

If $c \neq 0$ then a fixed point exists as the field of constant is algebraically closed. If $c = 0$, then if $d \neq a$ a fixed point exists. If $d = a \neq 0$, then the group is infinite unless $b = 0$, which correspond to the identity (the field is of characteristic zero). Hence the result. □

The irreducible equation

$$\Psi(\mathfrak{t}) = \frac{\psi(\mathfrak{t})}{\phi(\mathfrak{t})} = T, \tag{5}$$

has got

$$N = np$$

roots in an algebraic closure. We suppose that their list is given by $\mathfrak{t}_j$, where $0 \leq j \leq N-1$. Consider the equation

$$\frac{\psi(\eta)}{\phi(\eta)} = \frac{\psi(\eta_0)}{\phi(\eta_0)} = \frac{\psi(\eta_1)}{\phi(\eta_1)} = \cdots = \frac{\psi(\eta_{n-1})}{\phi(\eta_{n-1})}. \tag{6}$$

If η_j is one root it, then for any σ in its stabilizer S_j, η_j is a solution of the equation

$$\frac{\psi(\sigma.\eta)}{\phi(\sigma.\eta)} = \frac{\psi(\eta_j)}{\phi(\eta_j)}; \tag{7}$$

therefore η_j is root of equation (6) of multiplicity p, the order of the stabilizer. Moreover all the stabilizers are conjugated, therefore they have the same order. Thus we see that all the roots of equation (6) have multiplicity p and that equation (6) is the p-th power of a monic polynomial of degree n which we will call U. Let us try to take advantage of the polynomial U and of Ψ to parameterize a root x_i of an anharmonic h_n. We remind that the sum of the roots of an anharmonic h_n can be taken to be zero (by using an homography). We have the proposition

Proposition 7. *Consider one of the* $\mathfrak{t}$ *as defined by equation* (5) *and* η_j *the roots of* U. *Impose the condition*

$$x_i := P(\mathfrak{t}) \left[\frac{1}{\mathfrak{t} - \eta_i} + Q(\mathfrak{t}) \right]$$

and that

$$P(s(\mathfrak{t})) = \frac{ad-bc}{(c\mathfrak{t}+d)^2} P(\mathfrak{t})$$

with

$$s(z) = \frac{az+b}{cz+d},\ ad-cb \neq 0.$$

Then x_i *admits the parameterization*

$$x_i = \frac{\left[\dfrac{1}{\mathfrak{t}-\eta_i} - \dfrac{U'(\mathfrak{t})}{nU(\mathfrak{t})} \right]}{\Psi'(\mathfrak{t})}.$$

Proof. As

$$\sum_i x_i = 0,$$

one has

$$\sum_i x_i = P\left(nQ + \sum_i \frac{1}{\mathfrak{t} - \eta_i}\right) = 0.$$

But

$$\sum_i \frac{1}{\mathfrak{t} - \eta_i} = \frac{U'(\mathfrak{t})}{U(\mathfrak{t})};$$

hence

$$Q(\mathfrak{t}) = -\frac{U'(\mathfrak{t})}{nU(\mathfrak{t})}.$$

Now if P and P_1 are two rational functions satisfying the conditions of the proposition, then

$$\frac{P_1}{P}$$

is left invariant by all linear fractional transformations, therefore it is a rational function with numerical coefficients of $\Psi(\mathfrak{t})$. As the latter is supposed to be equal to $T \in \mathbb{F}$, $\dfrac{P_1}{P} \in \mathbb{F}$. P_1 differs from P only by a multiplicative factor belonging to $\mathbb{F}$. Suppressing this factor is equivalent by the below representation of x_i:

$$x_i := P(\mathfrak{t})\left[\frac{1}{\mathfrak{t} - \eta_i} + Q(\mathfrak{t})\right];$$

to multiply x_i by an element of $\mathbb{F}$. This does not change any of the fundamental properties of the anharmonics (the constance of the cross-ratio and the irreducibility). Therefore it suffices to get an arbitrary $P(\mathfrak{t})$ in order to solve the problem. Let Ψ' be the derivative of the absolute invariant with respect to $\mathfrak{t}$. As

$$\Psi\left(\frac{a\mathfrak{t} + b}{c\mathfrak{t} + d}\right) = \Psi(\mathfrak{t}),$$

then

$$\Psi'\left(\frac{a\mathfrak{t} + b}{c\mathfrak{t} + d}\right) = \Psi'(\mathfrak{t})\frac{\partial \mathfrak{t}}{\partial\left(\dfrac{a\mathfrak{t} + b}{c\mathfrak{t} + d}\right)} = \Psi'(\mathfrak{t})\frac{(c\mathfrak{t} + d)^2}{ad - bc}.$$

Now we just take

$$P(\mathfrak{t}) := \frac{1}{\Psi'(\mathfrak{t})}.$$

This completes the proof. □

One has therefore for expression of the x_i

$$x_i = \frac{\left[\dfrac{1}{\mathfrak{t} - \eta_i} - \dfrac{U'(\mathfrak{t})}{nU(\mathfrak{t})}\right]}{\Psi'(\mathfrak{t})} := \mathfrak{f}(\mathfrak{t}, \eta_i) \tag{8}$$

with

$$\Psi(\mathfrak{t}) = T \quad \text{and} \quad U(\eta_i) = 0.$$

We show that the general solution of the Riccati equation admits a representation in the form $u = \mathfrak{f}(\mathfrak{t}, C)$, C an arbitrary constant.

Lemma 8. *The general solution u of the given Riccati equation* (3) *can be written in the form*

$$u = \mathfrak{f}(\mathfrak{t}, C).$$

Proof. Take three solutions x_1, x_2, x_3 with respective representations

$$x_1 = \mathfrak{f}(\mathfrak{t}, \eta_1), \quad x_2 = \mathfrak{f}(\mathfrak{t}, \eta_2), \quad x_3 = \mathfrak{f}(\mathfrak{t}, \eta_3).$$

Define ζ:

$$u = \mathfrak{f}(\mathfrak{t}, \rho).$$

One knows that the cross-ratio of u, x_1, x_2, x_3, is a constant K_c . So

$$\frac{(u - x_1)(x_3 - x_1)}{(x_3 - x_2)(u - x_2)} = K_c.$$

But by very construction

$$\frac{(u - x_1)(x_3 - x_2)}{(x_3 - x_1)(u - x_2)} = \frac{(\rho - \eta_1)(\eta_3 - \eta_2)}{(\eta_3 - \eta_1)(\rho - \eta_2)} = K_c.$$

So ρ is a constant whose value depends on K_c. It is the arbitrary constant C. □

Now we have that

$$\Psi(\mathfrak{t}) = T$$

and

$$u = \mathfrak{f}(\mathfrak{t}, C).$$

This leads to

$$C = \mathfrak{t} - \left[u\Psi'(\mathfrak{t}) + \frac{U'(\mathfrak{t})}{nU(\mathfrak{t})} \right]^{-1}.$$

Taking the derivative $\dfrac{d}{d\mathfrak{t}}$, we get

$$\Psi' \frac{du}{d\mathfrak{t}} + u\Psi'' + \frac{d}{d\mathfrak{t}} \left(\frac{U'}{nU} \right) + \left(u\Psi' + \frac{U'}{nU} \right)^2 = 0. \tag{9}$$

This gives the form of the Riccati equation (2) associated to the algebraic equation (3) all of whose roots are solutions of the Riccati equation (2).

Remark 9. Elimination of $\mathfrak{t}$ between

$$T = \Psi(\mathfrak{t}) \quad \text{and} \quad x_i = \mathfrak{f}(\mathfrak{t}, \eta_i),$$

and setting the resulting equation equal to zero leads to the desired form of the anharmonic equation of the theorem (1). More precisely, the resulting equation is of degree N; also one sees that the construction does not depend of η_i. Moreover the stabilizer S_i of η_i (which is the stabilizer of x_i), fixes globally the two equations; therefore the equation we get, is the p-th power of a polynomial $F(x_i, T)$ (as any of its roots will have multiplicity p), the polynomial which we are looking for (after an eventual division by the leading coefficient). This monic polynomial of degree n is in $\mathbb{F}[x]$ and is irreducible as its p-th power does and its n roots are given by construction by the n solutions of the anharmonic h_n.

2.1. **Degree of the anharmonics.** We have seen that each anharmonic h_n, has a Galois group H of cardinal $N = np$, with p the cardinal of a stabilizer of a point (the stabilizers are conjugate) and $p|(n-1)$ or $p|(n-2)$. Moreover H is either cyclic, or dihedral, tetrahedral or octahedral or icosahedral. Also given any such group H, all its subgroups under the previous constraints are susceptible of being a stabilizer of a root of an anharmonic, as in the reasoning for the construction of an anharmonic, the latter were arbitrary.

We are going to use these specifications in order to determine the possible degrees n.

(1) First of all one remarks that if $p = 1$, H can be any of the finite subgroups of $PSL(2, \mathcal{C})$ and n assumes an arbitrary value ≥ 4, for the cyclic case; an arbitrary even value ≥ 4 for the dihedral case. It takes the value $n = 12$ for the tetrahedral case, the value $n = 24$ for the octahedral case and the value $n = 60$ for the icosahedral case.

(2) Next assume $p > 1$.

  (a) Then H can not be cyclic. This is because the group G acts transitively on the roots of h_n, so the stabilizers of the roots are conjugate. The group being cyclic, hence commutative, we see that all the stabilizers are cyclic subgroups and are the same: $H_{x_i} = H_{x_j}$ for $i \neq j$. Therefore H_{x_i} fixes all the roots of the anharmonic h_n, $n \geq 4$ and is different from the identity ($p > 1$); this is absurd because of the constance of the cross-ratio (any subgroup of H which fixes more than two roots, fixes all of them and H_{x_i} should be the identity).

  (b) If H is dihedral then $p = 2$; the case $p|(n-1)$ gives an odd n and $p|(n-2)$ an even n. To justify this assertion we remark that the dihedral group group of order $N = 2m$ is generated by two transformations Θ, ϵ with

$$\Theta^m = \mathrm{Id}; \quad \epsilon^2 = \mathrm{Id}, \quad \epsilon^{-1}\Theta\epsilon = \Theta^{-1}.$$

  It contains the $2m$ transformations

$$\Theta^i,\ i \in \{0, \cdots, m-1\} \text{ and } \epsilon\Theta^r,\ r \in \{0, \cdots, m-1\}.$$

  We claim that the stabilizer of a root x_i, H_{x_i} can not be one of Θ^i, $i \in \{0, \cdots, m-1\}$. The argument is based on the transitivity of the Galois group H, whigh gives that the stabilizers H_{x_i} are all conjugated.
  If H_{x_k} is of the form $\langle \Theta^l \rangle$ and is conjugated to H_{x_j} with $l \neq j$, via some $\epsilon\Theta^d$, then a quick computation gives

$$\Theta^{-d}\epsilon^{-1}\Theta^p\epsilon\Theta^d = \Theta^{-d}\Theta^{-1}\Theta^d = \Theta^{-1}.$$

  Therefore $H_{x_j} = \langle \Theta^{-1} \rangle$; thus because the operation of conjugation is an automorphism, one has $\sharp(H_{x_k}) = \sharp(H_{x_j}) = \sharp(\langle \Theta \rangle) = p = m$. Hence because $N = np = 2m$, one gets $n = 2$; absurd. So whenever $H_{x_k} = \langle \Theta^p \rangle$ has a conjugated stabilizer via an element of the form $\epsilon\Theta^d$, we are led to a contradiction.
  We must under this fact examine the other alternative which is: the remaining stabilizers are all conjugated to H_{x_p} via elements of the form Θ^d. The conclusion in this latter case results from the arguments given when H was cyclic. To summarize, in the dihedral case, the stabilizers are of the form $H_{x_i} = \langle \epsilon\Theta^i \rangle$ and $p = 2$.

  (c) In the tetrahedral case $N = 12 = np$. The tetrahedral group contains only elements of order 2 or 3. If $p = 2$, then $n = 6$. The only case which can happen also in the tetrahedral case is $p = 3$ and $n = 4$.

  (d) If the Galois group H is octahedral, then $N = 24 = np$. The octahedral group contains elements of order at most 4. If $p = 2$, then $n = 12$; $p = 3$ gives $n = 8$ and finally $p = 4$ corresponds to the value $n = 6$.

(e) Lastly the icosahedral has three cases; $p = 2$ and $n = 30$. There appears also the case $p = 3$, $n = 20$ and finally the case $n = 5$, $p = 12$. The icosahedral group contains as is well-known only substitution of order 2, 3 and 5.

These various cases according to $p = 1$ or $p > 1$ are summarized in the following tables

Group	n	p
Cyclic	n	1
Dihedral	n	1
Tetrahedral	12	1
Octahedral	24	1
Icosahedral	60	1

TABLE 1. The case $p = 1$.

Group	n	p
–	–	–
Dihedral	n	2
Tetrahedral	{4,6}	{3,2}
Octahedral	{2,3,4}	{12,8,6}
Icosahedral	{2,3,5}	{30,20,12}

TABLE 2. The case $p > 1$.

Example 10. Assume we are in the cyclic case, then $p = 1$, $n = N$, $\Psi_{cyc} = \mathfrak{t}^n = T$. Also $U(\eta) = \eta^n - K$, with $K \in \mathcal{C}$. One has

$$x = \mathfrak{f}(\mathfrak{t}, \eta) = \frac{1}{n\mathfrak{t}^{n-1}} \left(\frac{1}{\mathfrak{t} - \eta} - \frac{\mathfrak{t}^{n-1}}{\mathfrak{t}^n - K} \right);$$

this implies

$$\frac{\eta}{\mathfrak{t}} = 1 - \left(nTx + \frac{T}{T-1} \right), \, T \in \mathbb{F}.$$

This can be again written

$$\frac{\eta}{\mathfrak{t}} = x$$

as such a transformation preserves the fundamental properties of anharmonics. In conclusion we get

$$x^n = \frac{K}{T}.$$

3. Riccati equation, Darboux-Halphen System and Chazy equation

Our goal in this section is to show that Riccati equations are not exceptional in their property of satisfying algebraic equations. Indeed we are going to exhibit a third order polynomial, all of whose three roots satisfy a system of Darboux-Halphen type, which in the generic situation is irreducible (when the three solutions of the Darboux system are distinct). We will also consider a

remarkable irreducible third order polynomial whose three roots satisfy the Riccati equation. We fix once and for all a generic lattice in $\mathbb{C}$.

Let us introduce the basic definitions about modular forms. We recall there is a natural way to identify modular forms with some functions defined on lattices (two dimensional $\mathbb{Z}$-submodules of $\mathbb{C}$) in $\mathbb{C}$ having some invariance and homogeneity. We refer to [26] for details. Let $\mathfrak{R}$ be the set of lattices in the $\mathbb{R}$-vector space $\mathbb{C}$ and $\mathfrak{M} = \{(\omega_1, \omega_3) \in \mathbb{C}^{*2},\, \tau = \Im\frac{\omega_1}{\omega_3} > 0\}$. Let also $\mathfrak{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$. Then $\mathfrak{R}$ identifies to the quotient $\mathfrak{M}/SL(2,\mathbb{Z})$. Moreover $\mathbb{C}^*$ acts on $\mathfrak{R}$ and on $\mathfrak{M}$ with two more identifications

$$\mathfrak{M}/\mathbb{C}^* \approx \mathfrak{H}, \quad \mathfrak{R}/\mathbb{C}^* \approx \mathfrak{H}/PSL(2,\mathbb{Z}).$$

A function $F : \mathfrak{R} \mapsto \mathbb{C}$ is of weight n if $F(\lambda\Lambda) = \lambda^{-n}F(\Lambda)$ for all lattices Λ and all $\lambda \in \mathbb{C}^*$. For $(\omega_1, \omega_3) \in \mathfrak{M}$ and $\Lambda = \Lambda(\omega_1, \omega_3) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_3$, the lattice generated by ω_1 and ω_3, we write simply $F(\omega_1, \omega_3) = F(\Lambda)$ so that $F(\lambda\omega_1, \lambda\omega_3) = \lambda^{-n}F(\omega_1, \omega_3)$. Moreover $F(\omega_1, \omega_3)$ is invariant under the action of $SL(2,\mathbb{Z})$ on $\mathfrak{M}$ (as the lattice $\Lambda(\omega_1, \omega_3)$ is). This implies that there exists a function

$$f : \mathfrak{H} \mapsto \mathbb{C},\; F(\omega_1, \omega_3) = \omega_3^{-n} f\left(\frac{\omega_1}{\omega_3}\right).$$

The invariance of F under $SL(2,\mathbb{Z})$ means that $f(\tau) = (\gamma\tau + \delta)^{-n} f\left(\dfrac{\alpha\tau + \beta}{\gamma\tau + \delta}\right)$. For even n, we recover the classical invariance property of elliptic modular forms. Moreover as I_2 and $-I_2$ act in the same way, one can consider that it is the modular group $PSL(2,\mathbb{Z})$ which acts on the functions F.

On the other hand, one of the most general principle in elliptic function theory is that for any elliptic functions $\phi(u, \omega_1, \omega_3)$ of periods $2\omega_1, 2\omega_3$, the two functions

$$\begin{aligned} f(u) &= \omega_1 \frac{\partial\phi}{\partial\omega_1} + \omega_3 \frac{\partial\phi}{\partial\omega_3} + u\frac{\partial\phi}{\partial u} \\ g(u) &= \eta_1 \frac{\partial\phi}{\partial\omega_1} + \eta_3 \frac{\partial\phi}{\partial\omega_3} + \zeta(u)\frac{\partial\phi}{\partial u} \end{aligned}$$

are also elliptic with the same periods $2\omega_1, 2\omega_3$. We set $\omega_2 = \omega_1 + \omega_3$ and adopt the classical notations for Weierstrass elliptic functions

$$\wp(u) = \wp(u; \omega_1, \omega_3) = \frac{1}{u^2} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(u + 2m\omega_1 + 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2} \right). \tag{10}$$

Let

$$\zeta(u, \omega_1, \omega_3) := \frac{1}{u} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(u + 2m\omega_1 + 2n\omega_3)} + \frac{1}{(2m\omega_1 + 2n\omega_3)} + \frac{u}{(2m\omega_1 + 2n\omega_3)^2} \right).$$

One has

$$\zeta' = -\wp, \quad \eta_1 = \zeta(\omega_1), \quad \eta_3 = \zeta(\omega_3). \tag{11}$$

We set

$$\tau = \frac{\omega_3}{\omega_1},\; \Im\tau > 0, \quad q = e^{2i\pi\tau}$$

and

$$\begin{aligned} g_2 &= 60 \sum_{m^2+n^2 \neq 0} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, \\ g_3 &= 140 \sum_{m^2+n^2 \neq 0} \frac{1}{(2m\omega_1 + 2n\omega_3)^6}. \end{aligned} \tag{12}$$

As a result

$$\begin{aligned} -2\wp =&\omega_1 \frac{\partial \wp}{\partial \omega_1} + \omega_3 \frac{\partial \wp}{\partial \omega_3} + u \frac{\partial \wp}{\partial u} \\ -2\wp^2 + \frac{1}{3} g_2 =&\eta_1 \frac{\partial \phi}{\partial \omega_1} + \eta_3 \frac{\partial \phi}{\partial \omega_3} + \zeta(u) \frac{\partial \phi}{\partial u}. \end{aligned} \tag{13}$$

The constants g_2, g_3 are related to the classical Eisenstein series E_4 and E_6 by

$$\begin{aligned} E_4(\tau) =&12 \left(\frac{\omega_1}{\pi}\right)^4 g_2 = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} \\ E_6(\tau) =&216 \left(\frac{\omega_1}{\pi}\right)^6 g_3 = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} \end{aligned} \tag{14}$$

Set also

$$E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - \cdots .$$

We define

Definition 11. *Let Γ be a subgroup of finite index of the the modular group $SL(2,\mathbb{Z})$. A meromorphic (respectively holomorphic) function $f : \mathfrak{H} \mapsto \mathbb{C}$ is a weak meromorphic modular function of weight k with respect to Γ if*

$$f\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = (\gamma\tau + \delta)^n f(\tau)$$

for every $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$. If f is meromorphic (respectively holomorphic) at the cusps of Γ, it will be called a meromorphic (respectively holomorphic) modular form of weight n with respect to Γ.

These above Eisenstein series are modular forms of weight 4 and 6 respectively for the the subgroup $SL(2,\mathbb{Z})$. It is a fundamental result, that will be used later, that every modular form for $SL(2,\mathbb{Z})$ is uniquely expressible as a polynomial in E_4 and E_6 and the extension ring $\mathbb{C}[E_2, E_4, E_6]$ of $\mathbb{C}[E_4, E_6]$ is a differential ring. More precisely, the following basic relations of Ramanujan hold [24]:

$$\begin{aligned} \frac{1}{2i\pi} \frac{d}{d\tau} E_4 &= \frac{1}{3}(E_2 E_4 - E_6) \\ \frac{1}{2i\pi} \frac{d}{d\tau} E_6 &= \frac{1}{2}(E_2 E_6 - E_4^2) \\ \frac{1}{2i\pi} \frac{d}{d\tau} E_2 &= \frac{1}{12}(E_2^2 - E_4). \end{aligned} \tag{15}$$

In other words, the field $\left(\mathbb{C}\left(E_2, E_4, E_6\right), \frac{1}{2i\pi}\frac{d}{d\tau}\right)$ is a differential field. The subfield of constants is $\mathbb{C}$ (as it embedded in the field of meromorphic functions on $\mathfrak{H}$).

For the particular value $u = \omega_i$, $i = 1, 2, 3$, the equations (13) become

$$\begin{aligned} -2e_i &= \omega_1 \frac{\partial e_i}{\partial \omega_1} + \omega_3 \frac{\partial e_i}{\partial \omega_3}, \\ -2e_i^2 + \frac{1}{3} g_2 &= \eta_1 \frac{\partial e_i}{\partial \omega_1} + \eta_3 \frac{\partial e_i}{\partial \omega_3}. \end{aligned} \tag{16}$$

We will later deal with the two partial differential operators

$$D_2 = \eta_1 \frac{\partial}{\partial \omega_1} + \eta_3 \frac{\partial}{\partial \omega_3}.$$

Its importance lies in the fact that it converts ellipticity properties into differential relations for certain modular forms [15, 13, 11].

3.1. e_1, e_2, e_3 **and Darboux-Halphen system.**

Theorem 12. *For every $\tau, \Im\tau > 0$, the function $e_i(\tau) = \wp(\omega_i; 2\omega_1, 2\omega_3)$, $i = 1, 2, 3$, solves the nonlinear differential equation of Riccati type, with coefficients in $\mathbb{C}(E_2, E_4, E_6)$*

$$y' = \frac{i}{\pi}\Big(-y^2 + \frac{\pi^2}{3}E_2(\tau)y + \frac{2}{9}\pi^4 E_4(\tau)\Big)$$

Proof. We have seen in equation (16), that for the operator $D_2 = -2\eta_1 \frac{\partial}{\partial \omega_1} - 2\eta_3 \frac{\partial}{\partial \omega_3}$

$$D_2 e_i = 4e_i^2 - \frac{2}{3}g_2, \quad i = 1, 2, 3.$$

We consider the new independent variables $\omega_1,\ \tau = \dfrac{\omega_3}{\omega_1}$. The identities

$$e_k(2\omega_1, 2\omega_3) = (2\omega_1)^{-2} e_k(1, \tau), \quad g_k(2\omega_1, 2\omega_3) = (2\omega_1)^{-4} g_k(1, \tau),$$
$$\eta_1\omega_1 = \frac{\pi^2}{12}E_2(\tau), \quad \eta_2\omega_2 = \frac{\pi^2}{12}\tau^2 E_2(\tau) - \frac{i\pi\tau}{2}$$

and a straightforward calculation give the theorem. □

The partial differential operator D_2 can be replaced by a single differential operator. For $\omega = 2i\pi\tau$, e_i verifies

$$\frac{d}{d\omega}\left(\omega_1^2 e_i\right) = \frac{1}{\pi^2}\left(\frac{1}{3}g_2\omega_1^4 + \frac{\pi^2}{6}e_i\omega_1^2 E_2 - 2e_i^2\omega_1^4\right), \quad 1 \le i \le 3. \tag{17}$$

The system of differential equations (15) is now necessary to show that the system (17) is leads to a Darboux-Halphen system. Setting $t = \dfrac{4i}{\pi}\tau$ and

$$x = \frac{\pi^2}{12}E_2, \quad y = \frac{\pi^4}{12}E_4, \quad z = \frac{\pi^6}{216}E_6,$$

the system (15) becomes

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2}x^2 - \frac{1}{24}y \\ \frac{dy}{dt} &= 2xy - 3z \\ \frac{dz}{dt} &= 3xz - \frac{1}{6}y^2. \end{aligned} \tag{18}$$

With $X_k = \frac{\pi^2}{12}E_2 + \frac{1}{4}e_k, k = 1, 2, 3$, the equations (17) take the form

$$\begin{aligned} \frac{d}{dt}(X_1 + X_2) &= X_1 X_2 \\ \frac{d}{dt}(X_2 + X_3) &= X_2 X_3 \\ \frac{d}{dt}(X_3 + X_1) &= X_3 X_1, \end{aligned} \tag{19}$$

which is a Darboux-Halphen system. Moreover It is transformable into (18) by means of the substitutions:

$$
\begin{aligned}
x &= \frac{1}{3}\left(X_1 + X_2 + X_3\right) \\
y &= \frac{4}{3}\left(X_1^2 + X_2^2 + X_3^2 - X_1X_2 - X_2X_3 - X_3X_1\right) \\
z &= \frac{4}{27}\left(2X_1 - X_2 - X_3\right)\left(2X_2 - X_3 - X_1\right)\left(2X_3 - X_1 - X_2\right).
\end{aligned} \tag{20}
$$

3.2. **Chazy Equations.** A particular case of Chazy equation is the following third order differential equation

$$\gamma''' = 6\gamma\gamma'' - 9{\gamma'}^2,\ \gamma = \gamma(\tau). \tag{21}$$

It has the particular solution $\gamma(\tau)$, with $\gamma(\tau) = \sum_{n\geq 0} a_n q^n, q = e^{2i\pi\tau}$, unique up to the substitution $\tau \mapsto \tau + \tau_0$, $a_n \mapsto a_n e^{2in\tau_0}$. It is related to the Eisenstein E_2 as follows

$$\gamma(\tau) = -\frac{1}{6}(1 - 24q - 72q^2 - 96q^3 - 168q^4 - \cdots) = -\frac{1}{6}E_2(\tau).$$

This can be seen from (15) by eliminating E_4 and E_6. It is a remarkable fact [10] that if γ solves the Chazy equation, the three solutions $\omega_1(\tau)$, $\omega_2(\tau)$, $\omega_3(\tau)$ of the cubic equation

$$\omega^3 + \frac{3}{2}\gamma(\tau)\omega^2 + \frac{3}{2}\gamma'(\tau)\omega + \frac{1}{4}\gamma''(\tau) = 0 \tag{22}$$

are solutions of the Darboux-Halphen system:

$$
\begin{aligned}
\dot\omega_1 &= -\omega_1(\omega_2 + \omega_3) + \omega_2\omega_3 \\
\dot\omega_2 &= -\omega_2(\omega_1 + \omega_3) + \omega_1\omega_3 \\
\dot\omega_3 &= -\omega_3(\omega_1 + \omega_2) + \omega_1\omega_2.
\end{aligned} \tag{23}
$$

which is equivalent to (19) by setting $X_i = -2\omega_i$, $i = 1, 2, 3$.

Proof. One has

$$\omega_1 + \omega_2 + \omega_3 = -\frac{3}{2}\gamma(\tau),$$

$$\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 = \frac{3}{2}\gamma'(\tau)$$

and

$$\omega_1\omega_2\omega_3 = -\frac{1}{4}\gamma''(\tau).$$

Derivation of the three relations with respect to τ gives

$$\dot\omega_1 = -\dot\omega_2 - \dot\omega_3 - \frac{3}{2}\gamma'(\tau),$$

$$\left(-\dot\omega_2 - \dot\omega_3 - \frac{3}{2}\gamma'(\tau)\right)(\omega_2 + \omega_3) + \dot\omega_2\omega_1 + \dot\omega_3\omega_1 + \dot\omega_2\omega_3 + \dot\omega_3\omega_2 = \frac{3}{2}\gamma''(\tau)$$

and

$$\left(-\dot\omega_2 - \dot\omega_3 - \frac{3}{2}\gamma'(\tau)\right)\omega_2\omega_3 + \dot\omega_2\omega_1\omega_3 + \dot\omega_3\omega_1\omega_2 = -\frac{1}{4}\gamma'''(\tau).$$

This gives the system

$$\begin{aligned}\dot{\omega}_1 &= -\dot{\omega}_2 - \dot{\omega}_3 - \frac{3}{2}\gamma' \\ \dot{\omega}_2(\omega_1-\omega_2) + \dot{\omega}_3(\omega_1-\omega_3) &= \frac{3}{2}\gamma'' + \frac{3}{2}\gamma'(\omega_2+\omega_3) \\ \dot{\omega}_2\omega_3(\omega_1-\omega_2) + \dot{\omega}_3\omega_2(\omega_1-\omega_3) &= -\frac{1}{4}\gamma''' + \frac{3}{2}\gamma'\omega_2\omega_3\end{aligned}$$

Set $\Delta = (\omega_1-\omega_2)(\omega_1-\omega_3)(\omega_2-\omega_3)$; then suppose the three roots are all distinct; applying the Cramer formula one gets for the value of $\dot{\omega}_2$, the following

$$\dot{\omega}_2 = \frac{\frac{1}{4}\gamma'''(\omega_1-\omega_3) + \frac{3}{2}\gamma''\omega_2(\omega_1-\omega_3) + \frac{3}{2}\gamma'\omega_2^2(\omega_1-\omega_3)}{\Delta}.$$

Using the fact that γ satisfies the Chazy equation

$$\gamma''' = 6\gamma\gamma'' - 9{\gamma'}^2,\ \gamma = \gamma(\tau)$$

and the expressions of γ' and γ'' in terms of ω_1, ω_2 and ω_3, one gets after simplification the value of $\dot{\omega}_2$

$$\dot{\omega}_2 = -\omega_2(\omega_1+\omega_3) + \omega_1\omega_3.$$

The values of $\dot{\omega}_1$ and $\dot{\omega}_3$ follow in the same manner. □

Remark 13. Let $\Gamma = PSL(2,\mathbb{Z})$ be the modular group and $\Gamma(2)$ be its principal congruence group of level 2. The corresponding Riemann surfaces $\mathfrak{H}^\star/\Gamma = (\mathfrak{H}\cup\{\infty\})/\Gamma$ and $\mathfrak{H}^\star/\Gamma(2) = (\mathfrak{H}\cup\{\infty,0,1\})/\Gamma$, after suitable compactification, (adding cusps) have genus 0 and (after a suitable normalization) elliptic fixed points of orders 2 and 3 at i and $e^{i\frac{2\pi}{3}}$ for the case of Γ. Denote by J and λ holomorphic mappings which establish isomorphisms $H^\star/\Gamma \cong \mathbf{P}^1$ and $H^\star/\Gamma(2) \cong \mathbf{P}^1$. These "Hauptmodules" in the terminology of Klein, are modular functions for Γ and $\Gamma(2)$. One can express the general solutions of the Darboux-Halphen system and the Chazy equation in terms of these functions. We have [20, 21, 28] that the transformation rule

$$\omega^i(\tau) \to \frac{1}{(c\tau+d)^2}\omega_i\left(\frac{a\tau+b}{c\tau+d}\right) + \frac{c}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in PSL(2,\mathbb{C})$$

together with the formulas

$$\begin{aligned}\omega_1 &= -\frac{1}{2}\frac{d}{d\tau}\log\frac{\lambda'}{\lambda} \\ \omega_2 &= -\frac{1}{2}\frac{d}{d\tau}\log\frac{\lambda'}{\lambda-1} \\ \omega_3 &= -\frac{1}{2}\frac{d}{d\tau}\log\frac{\lambda'}{\lambda(\lambda-1)}\end{aligned}$$

provide the general solution to the Darboux-Halphen system. For the case of the Chazy equation we have that the transformation rule

$$\gamma \to \frac{1}{(c\tau+d)^2}\gamma\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{2c}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in PSL(2,\mathbb{C})$$

and the formula

$$\gamma = \frac{1}{6}\log\frac{(J')^6}{J^4(J-1)^3}$$

provide the general solution for the Chazy equation.

When the three solutions of the Darboux-Halphen system are not distinct, for example when $\omega_2 = \omega_3$, say, then

$$\omega_1 = \frac{c}{c\tau + d} - \frac{a}{(c\tau + d)^2}, \quad \omega_2 = \omega_3 = \frac{c}{c\tau + d}.$$

When they are all equal they are given by

$$\omega_1 = \omega_2 = \omega_3 = \frac{1}{\tau - \tau_0}$$

with τ_0 an arbitrary constant and modulo the transformation property.

The Chazy equation admits also the rational solution

$$\gamma = -\frac{2c}{c\tau + d} + \frac{2}{3}\frac{a}{(c\tau + d)^2}.$$

Remark 14. The Darboux-Halphen system arose historically in 1878 in the study by Darboux of the existence in Euclidean space of a one-parameter family of surfaces orthogonal to two arbitrary given independent families of parallel surfaces (such a family is necessarily quadratic and ruled), and it was solved in 1881 by Halphen. This equation occurs in the Bianchi IX cosmological model and also appears as a special reduction (obtained by change of parameters) of the self-dual Yang-Mills equation. See [28].

3.3. **The degree three polynomials for the Darboux-Halphen system and the Riccati equation.** We remind that the Weirstrass function

$$\wp(u) = \wp(u; \omega_1, \omega_3) = \frac{1}{u^2} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(u + 2m\omega_1 + 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2} \right)$$

satisfies the equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 \tag{24}$$

But we have

$$\wp'\left(\frac{1}{2}\right) = \wp'\left(\frac{\tau}{2}\right) = \wp'\left(\frac{\tau+1}{2}\right) = 0; \tag{25}$$

therefore with the previous notation of theorem (12), we see that e_1, e_2, e_3, solve the equation

$$4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3) = 0. \tag{26}$$

Remark 15. One can show that e_1, e_2, e_3 are modular forms of weight 2 for $\Gamma(2)$ and that they are distinct as the discriminant of the non singular elliptic curve $Y^2 = 4X^3 - g_2X - g_3$ is the discriminant of the degree three polynomial $4X^3 - g_2X - g_3$ up to non zero constant factor.

We have the well-known fact

Proposition 16. *For all* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$, *we have*

$$E_2\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = (\gamma\tau + \delta)^2 E_2(\tau) + 6\frac{\gamma}{i\pi}(\gamma\tau + \delta).$$

Now as e_1, e_2 and e_3 satisfy the identity

$$e_1 + e_2 + e_3 = 0,$$

one easily sees that the e_i can not all belong to $\mathbb{C}(E_2, E_4, E_6)$ as the functions E_2, E_4 and E_6 are algebraically independent over $\mathbb{C}$. Our goal is to show that the polynomial

$$X^3 - \frac{g_2}{4}X - \frac{g_3}{4}$$

whose three roots are the $e_i \in M_2(\Gamma(2))$ is irreducible on the differential field $\mathbb{C}(E_2, E_4, E_6)$. g_2 is modular of weight 4 and g_3 modular of weight 6 for $SL(2,\mathbb{Z})$. Now we know that the e_i are modular only of weight 2 for $\Gamma(2)$; to show that

$$X^3 - \frac{g_2}{4}X - \frac{g_3}{4}$$

is irreducible on the field $\mathbb{C}(E_2, E_4, E_6)$, it suffices to show that it is integral on $\mathbb{C}[E_2, E_4, E_6]$ and then use Gauss irreducibility criterion. This is equivalent to the fact that none of the e_i belongs to the latter field. To see this is true, we assume for instance it were the case that e_1 belonged to $\mathbb{C}[E_2, E_4, E_6]$. e_1 is modular of weight 2 for $\Gamma(2)$, using the transformation rule for E_2 applied to the matrix $g = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (proposition 16), one sees that E_2 is not modular for $\Gamma(2)$. As the weights of E_4 and E_6 are 4 and 6, e_1 can not be a polynomial in them. Thus one gets a contradiction. Therefore we have the proposition

Proposition 17. *The solutions e_i of the Riccati equation of theorem 12 are the roots of the irreducible degree three equation $X^3 - \frac{g_2}{4}X - \frac{g_3}{4}$ on the differential field $\left(\mathbb{C}\left(E_2, E_4, E_6\right), \frac{1}{2i\pi}\frac{d}{d\tau}\right)$. The Galois group of the equation is thus S_3.*

We want now to show that in the generic case (all three solutions distinct) the solutions of the Darboux-Halphen system are algebraic over the differential field $\left(\mathbb{C}(\gamma, \gamma', \gamma''), \frac{d}{d\tau}\right) \subset \left(\mathbb{C}(E_2, E_4, E_6), \frac{d}{d\tau}\right)$; with the solution of the Chazy equation also considered in the generic case. For that we introduce the concept of quasimodular forms due to Zagier and Kaneko [16].

Definition 18. *A (meromorphic) quasimodular form of weight k and depth p on Γ is a meromorphic function f on $\mathfrak{H}$ such that*

$$(27)\qquad \begin{aligned} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{i=0}^{p} f_i(z)\left(\frac{c}{c\tau + d}\right)^i, \\ z \in \mathfrak{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \end{aligned}$$

and where the f_i are meromorphic on $\mathfrak{H}$ with moderate growth at the cusps. The space of quasimodular forms of weight k and depth p on Γ is denoted by $\widetilde{M}_k^{(\leq p)} = \widetilde{M}_k^{(\leq p)}(\Gamma)$. The prototype of quasimodular form is the Eisenstein series E_2 that is of weight 2 and depth 1.

The following summarizes the properties of quasimodular forms

Proposition 19 ([16])**.** *Let Γ be a modular subgroup and let k and p be non negative integers*

- *The space of quasimodular forms on Γ is closed under differentiation*

$$D(\widetilde{M}_k^{(\leq p)}) \subset \widetilde{M}_{k+2}^{(\leq p+1)}$$

with $D = \frac{1}{2i\pi}\frac{d}{d\tau} - \frac{k}{12}E_2$.

- *Every quasimodular form on Γ is a polynomial in E_2 with modular forms as coefficients. More precisely we have*

$$\widetilde{M}_k^{(\leq p)} = \oplus_{r=0}^{p} M_{k-2r}(\Gamma).E_2^r$$

for all k, $p \geq 0$, where $M_j(\Gamma)$ denotes the space of weight j modular forms on Γ.

- *Every quasimodular form on Γ can be written uniquely as a linear combination of derivatives of modular forms and of E_2. More precisely we have*

$$\widetilde{M}_k^{(\leq p)} = \begin{cases} \oplus_{r=0}^{p} D^r(M_{k-2r}(\Gamma)) & \text{if } p < k/2, \\ \oplus_{r=0}^{k/2-1} D^r(M_{k-2r}(\Gamma)) \oplus \mathbb{C}D^{k/2-1}E_2 & \text{if } p \geq k/2. \end{cases}$$

In the generic situation, the functions $f_1 = \dfrac{\lambda'}{\lambda}$, $f_2 = \dfrac{\lambda'}{\lambda - 1}$ and $f_3 = \dfrac{\lambda'}{\lambda(\lambda-1)}$ (remark 13) are modular of weight 2 for $\Gamma(2)$. Therefore their logarithmic derivatives, hence the ω_i are quasimodular forms of weight 2 and depth 1 for $\Gamma(2)$ (just take the derivative of the modular relation). Moreover $E_2 = \dfrac{1}{2i\pi}\dfrac{\Delta'}{\Delta}$, with $\Delta = q\prod_{n\geq 1}(1-q^n)^{24}$ the modular discriminant of weight 12. Thus taking into account the previous theorem [16], one sees that none of $\dfrac{f_i'}{f_i}$ can belong to $\mathbb{C}E_2$ as this would lead to a relation

$$\frac{f_i'}{f_i} = c\frac{\Delta'}{\Delta}$$

with $c \in \mathbb{C}^\star$. This is a contradiction as f_i is of weight 2 and Δ of weight 12 for $\Gamma(2)$. In other words one has a non trivial relation

$$\frac{f_i'}{f_i} = g_i + c_i E_2$$

with $g_i \in M_2(\Gamma(2))$ and $c_i \neq 0$. One immediately sees, using a similar approach to the case of proposition (16), that $\dfrac{f_i'}{f_i} \notin C[E_2, E_4, E_6]$ as this latter contains no modular form of weight 2 for $\Gamma(2)$. We reach therefore a similar conclusion to the case of proposition 16 that is

Proposition 20. *If the solutions of the Darboux-Halphen system are generic* (*ie all distinct*) *and also if the solution of the Chazy equation is generic* (*not a rational function*) *then the polynomial*

$$\omega^3 + \frac{3}{2}\gamma(\tau)\omega^2 + \frac{3}{2}\gamma'(\tau)\omega + \frac{1}{4}\gamma''(\tau)$$

is irreducible on the differential field $\left(\mathbb{C}(\gamma, \gamma', \gamma''), \dfrac{d}{d\tau}\right)$. Therefore the extension $\mathbb{C}(\omega)/\mathbb{C}(\gamma, \gamma', \gamma'')$ is galoisian with Galois group S_3.

4. Algebraic solutions of the Riccati equation on $\mathbb{C}(\wp(z), \wp'(z))$: Another approach

In this section we investigate the following question: for which potential q the Riccati equation

$$u' + u^2 = q$$

admit an algebraic solution over the field $\mathbb{C}(\wp(z), \wp'(z))$. We will discover this is given by the anharmonic Weirstrass function $\wp_0$ corresponding to the case $g_2 = 0$.

First of all we recall some notions from classical invariant theory. Let K be a field of characteristic zero. A binary form is by definition a homogeneous polynomial

$$f(x, y) = a_0x^n + \binom{n}{1}a_1x^{n-1}y + \binom{n}{2}a_2x^{n-2}y^2 + \cdots + a_ny^n$$

with coefficients in K, n is the degree of the form f. Another used notation is

$$f(x,y) = (a_0, a_1, \cdots a_n)(x,y)^n.$$

The set V_n of binary forms of degree n is a K-vector space of dimension n and can be identified with the space of polynomials $K[a_0, a_1, \cdots a_n]$ on which the groups $GL(2,K)$ acts in the following way:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,K),\ f \in V_n,\ \ (gf)(x,y) = f(ax+by, cx+dy).$$

We introduce the following important differential operator

$$\Omega := \frac{\partial^2}{\partial x \partial y'} - \frac{\partial^2}{\partial x' \partial y}.$$

It is known as Cayley Omega process.

For two given binary forms $Q(x,y), R(x,y)$, their transvectant of degree r is the function

$$(Q,R)^r = \sum_{i=0}^{r} (-1)^i \frac{\partial^r Q}{\partial^{r-i} x \partial^i y} \frac{\partial^r R}{\partial^i x \partial^{r-i} y}.$$

The transvectant of degree r can be obtained from the Cayley Omega process using the following formula

$$(Q,R)^r = \Omega^r \{Q(x,y), R(x',y')\}_{|x'=x,\, y'=y}.$$

For example

$$(Q,R)^{(1)} = Q_x R_y - Q_y R_x, \quad (Q,R)^{(2)} = Q_{xx} R_{yy} - 2Q_{xy} R_{xy} + Q_{yy} R_{xx}.$$

For polynomials of one variable F, we consider the projective coordinate $p = \dfrac{x}{y}$ and define the new polynomial $Q(x,y) = y^n F(\dfrac{x}{y})$. It follows that if F, G are polynomials of degrees n, m respectively, the r^{th} transvectant of F, G is, for $r \leq \min(m,n)$

$$(F,G)^r = \sum_{i=0}^{r} (-1)^k \binom{r}{k} \frac{(m-k)!}{(m-r)!} \frac{(n-r+k)!}{(n-r)!} \frac{d^{r-k} F}{dp^{r-k}} \frac{d^k G}{dp^k}.$$

It is of degree $\leq m + n - 2r$. Few examples are given by

$$\begin{aligned} (F,G)^0 &= FG, \\ (F,G)^1 &= mF_p G - nFG_p, \\ (F,G)^2 &= m(m-1)F_{pp} G - 2(m-1)(n-1)F_p G_p + n(n-1)FG_{pp}. \end{aligned}$$

and the Hessian

$$H(F) = \frac{1}{2}(F,G)^2 = n(n-1)\left(FF_{pp} - \frac{n-1}{n} F_p^2\right).$$

We need in the sequel the fourth transvectant

$$(F,F)^4 = 2(m-3)(m-2)\left(m(m-1)F^{(4)}F - 4(m-3)(m-1)F^{(3)}F' + 3(m-3)(m-2)F''^2\right)$$

In particular the fourth order differential equation corresponding to the vanishing of the fourth transvectant is

$$m(m-1)F^{(4)}F - 4(m-3)(m-1)F^{(3)}F' + 3(m-3)(m-2)F''^2 = 0. \tag{28}$$

Following [22] the equation (28) can be reduced to generalized Chazy equation which is a third order differential equation. For if R is given by $-mR = \frac{F'}{F}$, then an easy calculation gives

$$\begin{aligned} F' &= -nRF \\ F'' &= \left(-nR' + n^2R^2\right) F \\ F^{(3)} &= \left(-nR'' + 3n^2RR' - n^3R^3\right) F \\ F^{(4)} &= \left(-nR^{(3)} + 3n^2{R'}^2 + 4n^2RR'' - 6n^3R'R^2 + n^4R^4\right) F. \end{aligned} \tag{29}$$

So that $(F, F)^4 = 0$ becomes the generalized Chazy equation for R

$$R^{(3)} - 12RR' + 18{R'}^2 = \frac{6n^2}{n-1}\left(R' - R^2\right)^2.$$

On the other hand the coefficients of the fourth transvectant $(f, f)^4$ can be computed recursively from those of the binary form f. For $f(x, y) = (a_0, a_1, a_2, \cdots, a_n)(x, y)^n$ we have

$$\frac{1}{2}(f, f)^4(x, y) = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m)(x, y)^m,\ m \leq 2(n-4)$$

with

$$\begin{aligned} \alpha_0 &= a_0a_4 - 4a_1a_3 + 3a_2^2 \\ m_1\alpha_1 &= (n-4)(a_0a_5 - 3a_1a_4 + 2a_2a_3) \\ m_2\alpha_2 &= \frac{(n-4)(n-5)}{2}(a_0a_6 - 4a_1a_5 + 7a_2a_4 - 4a_3^2) + (n-4)^2(a_1a_5 - 4a_2a_3 + 2a_3^2) \\ \cdots &= \cdots \\ \alpha_m &= a_na_{n-4} - 4a_{n-1}a_{n-3} + 3a_{n-2}^2 \\ m_{m-1}\alpha_{m-1} &= (n-4)(a_na_{n-5} - 3a_{n-1}a_{n-4} + 2a_{n-2}a_{n-3}). \end{aligned} \tag{30}$$

More generally according to r is even or odd:

$$m_r\alpha_r = \sum_0^{\frac{r}{2}} p_{r,s}P_{r,s}; \quad m_r\alpha_r = \sum_0^{\frac{r-1}{2}} p_{r,s}P_{r,s}$$

where

$$m_r = \binom{m}{r}, \quad p_{r,s} = \binom{n-4}{s}\binom{n-4}{r-s}, \tag{31}$$

and

$$P_{r,s} = a_sa_{r-s+4} - 4a_{s+1}a_{r-s+3} + 6a_{s+2}a_{r-s+2} - 4a_{s+3}a_{r-s+1} + a_{s+4}a_{r-s}. \tag{32}$$

We now define on the ring $\mathbb{C}(\wp(z), \wp'(z))[u]$, the following differential operator, which will play a important role in the sequel

$$\begin{aligned} X : \mathbb{C}(\wp(z), \wp'(z))[u] &\longrightarrow \mathbb{C}(\wp(z), \wp'(z))[u] \\ f &\longmapsto \frac{\partial f}{\partial z} + (q - u^2)\frac{\partial f}{\partial u}. \end{aligned} \tag{33}$$

Let $\Phi(u) \in \mathbb{C}(\wp(z), \wp'(z))[u]$ be the irreducible polynomial of minimal degree n satisfied by a solution of the Riccati equation: $u' + u^2 = q$. The assumption of minimality means the irreducible polynomial of least degree satisfied by an eventual solution of the Riccati equation.

$$\Phi(u) \ = \ u^n + \frac{n}{1!}a_1u^{n-1} + \frac{n(n-1)}{2!}a_2u^{n-2} + \ldots + a_n \tag{34}$$

then

$$X(\Phi) = n(a_1 - u)\Phi \tag{35}$$

We define

Definition 21. *Let K be a field of characteristic zero, $K[X_1 \cdots, X_n]$ a ring of polynomials over it and D a derivation on $K[X_1, \cdots, X_n]$. A non-zero element P of $K[X_1, \cdots, X_n]$ is a called a Darboux polynomial for D if P is an eigenvector of D for its action on $K[X_1, \cdots, X_n]$, namely there exists $R \in K[X_1, \cdots, X_n]$:*

$$D(P) = RP.$$

When $R \equiv 0$ P is called a first integral. The eigenvalue is called a Darboux cofactor. Two Darboux polynomials having the same cofactor give rise to a first integral.

So $\Phi(u)$ is a Darboux polynomial for the mentioned derivation, ie a non trivial polynomial eigenvector of the derivation

$$X = \frac{\partial}{\partial z} + (q - u^2)\frac{\partial}{\partial u}.$$

Remark 22. It is perhaps worthwhile to observe that in general, an algebraic function f on a differential field (L, δ) of characteristic zero verifies a linear differential equation with coefficients in L. In fact if $f^n + a_1 f^{n-1} + \cdots + a_n = 0, a_i \in L$, $1 \le i \le n$ is the minimal polynomial of f over L, then necessarily $nf^{n-1} + (n-1)a_1 f^{n-2} + \cdots + a_{n-1} \neq 0$ and $f' \in L(f)$. Hence f'', $f^{(3)}, \cdots \in L(f)$. The dimension of the the L-vector space $L(f)$ is n and therefore the vectors $f^{(n)}$, $f^{(n-1)}, \cdots, f$ of $L(f)$ are linearly dependent over the field L, that is to say f is a solution of an homogeneous linear differential equation.

Remark 23. As our base field $K = \mathbb{C}(\wp(z), \wp'(z))$ is a field of meromorphic functions, the existence of a non-trivial first integral for X, $P(u)$, is equivalent to $P(u)$ being constant on the solutions. Therefore the general solution of our Riccati equation is given in this case by a relation

$$P(u) = \mathfrak{C}$$

$\mathfrak{C}$ an arbitrary constant belonging to $\mathbb{C}$. Moreover two first integrals for X can not be functionally independent as we know from the Cauchy theorem for differential equations, that the general solution of a first order equation depend only on one arbitrary parameter.

We have the following well-known lemma

Lemma 24 ([8])**.** *A non trivial polynomial $P(u)$ in $\mathbb{C}(\wp(z), \wp'(z))[u]$ is a Darboux polynomial for the derivation $\dfrac{\partial}{\partial z} + (q - u^2)\dfrac{\partial}{\partial u}$ if and only if all its roots are solutions of the Riccati equation $\dfrac{du}{dz} + u^2 = q$.*

Using equation (35), one sees that the coefficients of $\Phi(u)$ verify the following relations

$$\begin{aligned}
(n-1)a_2 &= na_1^2 - a_1' - q \\
(n-2)a_3 &= na_1a_2 - a_2' - 2a_1q \\
(n-3)a_4 &= na_1a_3 - a_3' - 3a_2q \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
(n-k)a_{k+1} &= na_1a_k - a_k' - ka_{k-1}q.
\end{aligned} \tag{36}$$

This shows at least that if it happens that a_1, q belong to some differential subring $L \subset K$ then so do the other coefficients a_k.
We consider the homogeneous polynomial

$$\begin{aligned} K(u_1, u_2) &= u_2^n \Phi(\frac{u_1}{u_2}) \\ &= \sum_{k=0}^{n} C_n^k a_k u_1^{n-k} u_2^k. \end{aligned}$$

and its Hessian

$$\begin{vmatrix} \dfrac{\partial^2 K}{\partial u_1^2} & \dfrac{\partial^2 K}{\partial u_1 \partial u_2} \\ \dfrac{\partial^2 K}{\partial u_1 \partial u_2} & \dfrac{\partial^2 K}{\partial u_2^2} \end{vmatrix} = (n-1) u_2^{2n-4} \left(n\Phi\Phi'' - (n-1)\Phi'^2 \right).$$

The polynomial $H = n\Phi\Phi'' - (n-1)\Phi'^2$ $(' = \frac{\partial}{\partial u})$ is of degree $\le 2(n-2)$. For latter use we have the following important property

Lemma 25. *Let K be field of characteristic zero and $\Phi(u) = u^n + \frac{n}{1!} a_1 u^{n-1} + \frac{n(n-1)}{2!} a_2 u^{n-2} + \ldots + a_n$ an irreducible polynomial then $H(\Phi) \not\equiv 0$.*

Moreover the partial differential operator X is a derivation

$$X(uv) = uX(v) + vX(u)$$

with the following commutation property

$$[X, \frac{\partial}{\partial u}] = 2u \frac{\partial}{\partial u}.$$

Hence

$$\begin{aligned} X(\Phi') &= -n\Phi + (na_1 - nu + 2u)\Phi' \\ X(\Phi'') &= -2(n-1)\Phi' + (na_1 - nu + 4u)\Phi'' \\ X(\Phi\Phi'') &= -2(n-1)\Phi\Phi' + (na_1 - nu + 2u)\Phi\Phi'' \\ X(\Phi'^2) &= 2\Phi' \left(-n\Phi + (na_1 - nu + 2u)\Phi' \right). \end{aligned}$$

and

$$X(H) = 2(na_1 - nu + 2u)H. \tag{37}$$

The Jacobian is

$$\begin{vmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial u} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial u} \end{vmatrix} = \begin{vmatrix} X(\Phi) & \Phi'_u \\ X(H) & H'_u \end{vmatrix} = na_1(\Phi H' - 2H\Phi') - u\Omega.$$

with $\Omega = n\Phi H' - 2(n-2)H\Phi'$. It is easily seen that

$$\begin{aligned} X(H') &= -2(n-2)H + 2(na_1 - nu + 3u)H' \\ X(\Phi H') &= -2(n-2)\Phi H + 2(na_1 - nu + 3u)\Phi H' + n(a_1 - u)\Phi H' \\ X(H\Phi') &= -n\Phi H + 3(na_1 - nu + 2u)\Phi' H. \end{aligned}$$

from which we obtain

$$X(\Omega) = 3(na_1 - nu + 2u)\Omega \tag{38}$$

and finally

$$3X(H)\Omega = 2HX\Omega$$

Or

$$\frac{\Omega^2}{H^3} = \Gamma. \tag{39}$$

where Γ is an arbitrary constant in $\mathbb{C}$ (see remark 23). Now we are going to exhibit another constant. We consider the Jacobian of Φ and Ω

$$\begin{vmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial u} \\ \frac{\partial \Omega}{\partial x} & \frac{\partial \Omega}{\partial u} \end{vmatrix} = \begin{vmatrix} X(\Phi) & \Phi'_u \\ X(\Omega) & \Omega'_u \end{vmatrix} = na_1(\Phi\Omega' - 3\Omega\Phi') - u\Omega_1.$$

with $\Omega_1 = n\Phi\Omega' - 3(n-2)\Omega\Phi'$.

In a similar way as before, we have

$$\begin{aligned} X(\Omega') &= -3(n-2)\Omega + (3na_1 - 3nu + 8u)\Omega' \\ X(\Phi\Omega') &= -3(n-2)\Phi\Omega + 4(na_1 - nu + 2u)\Phi\Omega' \\ X(\Omega\Phi') &= -n\Phi\Omega + 4(na_1 - nu + 2u)\Phi' \end{aligned}$$

so that

$$X(\Omega_1) = 4(na_1 - nu + 2u)\Omega_1. \tag{40}$$

The two identities (37), (40) give

$$2\Omega_1 X(H) = X(\Omega_1)H$$

that is, with another constant a

$$\frac{\Omega_1}{H^2} = a.$$

As we said before in remark 23 two non trivial first integrals are functionally dependent and this should force differential relation for our $\Phi(u)$. Indeed one has the following

$$\begin{aligned} H &= n\Phi\Phi'' - (n-1){\Phi'}^2 \\ H' &= n\Phi\Phi''' - (n-2)\Phi'\Phi'' \\ \Omega &= n\Phi H' - 2(n-2)H\Phi' \\ &= n^2\Phi^2\Phi''' - 3n(n-2)\Phi\Phi'\Phi'' + 2(n-1)(n-2){\Phi'}^3 \\ \Omega' &= -n(n-6)\Phi\Phi'\Phi''' + 3(n-2)^2{\Phi'}^2\Phi'' - 3n(n-2)\Phi{\Phi''}^2 + n^2\Phi^2\Phi'''' \\ \Omega_1 &= -4n^2(n-3)\Phi^2\Phi'\Phi''' - 3n^2(n-2)\Phi^2{\Phi''}^2 + n^3\Phi^3\Phi'''' + 12n(n-2)^2\Phi{\Phi'}^2\Phi'' \\ &- 6(n-1)(n-2)^2{\Phi'}^4 \\ aH^2 &= a\left(n^2\Phi^2{\Phi''}^2 - 2n(n-1)\Phi{\Phi'}^2\Phi'' + (n-1)^2{\Phi'}^4\right). \end{aligned}$$

In conclusion splitting the two members of the equality $\Omega_1 = aH_2$ into monomials involving only powers of Φ' on one side and the rest on the other side, one sees that Φ should divide Φ' unless $a(n-1)^2 = -6(n-1)(n-2)^2$ or $a = -6\dfrac{(n-2)^2}{n-1}$. Moreover we have the divisibility of $\Omega_1 - aH^2$

by Φ^2 giving the vanishing of the fourth transvectant

$$\tau_4(\Phi) := \frac{n-1}{n^2}\frac{\Omega_1 - aH^2}{\Phi^2} \tag{41}$$
$$= n(n-1)\Phi\Phi^{(4)} - 4(n-1)(n-3)\Phi^{'}\Phi^{'''} + 3(n-2)(n-3)\Phi^{''2}$$
$$= 0.$$

Set $L(u) = -\frac{\partial}{\partial u}\frac{\Omega^2}{H^3}$. It is a general fact that because $\frac{\Omega^2}{H^3}$ is a first integral of the Riccati equation and $[X, \frac{\partial}{\partial u}] = 2u\frac{\partial}{\partial u}$ that the following relation holds

$$X(L(u)) = -2uL(u). \tag{42}$$

So $L_1(u) = L(u)H^6$ is a Darboux polynomial. Computation gives

$$L_1(u) = \Omega(2H\Omega' - 3\Omega H')H^2 := \omega\Xi H^2.$$

$\Xi = 2H\Omega' - 3\Omega H'$. We want to establish a relation between Ξ and Φ which will enable us to give another first integral of the considered Riccati equation and show that $a_1 = 0$. First we calculate $\Xi\Phi$. One has by very definition

$$\Xi\Phi = 2H\Phi\Omega' - 3\Omega\Phi H'.$$

But

$$\Phi H' = \frac{1}{n}\left(\Omega + 2(n-2)H\Phi'\right),$$

so

$$-\frac{n}{3}\Xi\Phi = -\frac{2n}{3}H\Phi\Omega' + \Omega\left(\Omega + 2(n-2)H\Omega'\right) = \Omega^2 - \frac{2}{3}H\Omega_1.$$

As $\Omega_1 = aH^2$, with a the previously given value, one gets

$$-\frac{n}{3}\Xi\Phi = \Omega^2 - \alpha H^3,\ \alpha = -4\frac{(n-2)^2}{n-1}.$$

We remark that with this value of α, Φ^3 is a factor of $\Omega^2 - \alpha H^3$.

Let us take the derivation of the previous relation with respect to u; this gives

$$-\frac{n}{3}\left(\Xi\Phi' + \Phi\Xi'\right) = 2\Omega\Omega' - 3\alpha H^2 H'.$$

One sees that

$$\Omega' = \frac{\Xi + 3\Omega H'}{2H};$$

there we obtain the equation

$$\begin{aligned}
-\frac{n}{3}\left(\Xi\Phi' + \Phi\Xi'\right) &= 2\Omega\frac{\Xi + 3\Omega H'}{2H} - 3\alpha H^2 H' \\
&= \Omega\frac{\Xi + 3\Omega H'}{H} - 3\alpha H^2 H' \\
&= \frac{\Omega\Xi}{H} + \frac{3H'}{H}\left(\Omega^2 - \alpha H^3\right) \\
&= \frac{\Xi}{H}\left(\Omega - n\Phi H'\right) \\
&= -2(n-2)\Xi\Phi'.
\end{aligned}$$

Thus

$$\Phi\Xi' = \frac{6(n-2)}{n}\Phi'\Xi - \Phi'\Theta$$

and this gives finally

$$\Phi\Xi' = \left(5 - \frac{12}{n}\right)\Phi'\Xi. \tag{43}$$

It follows that

$$\Phi\Xi^{'} \;=\; \left(5 - \frac{12}{n}\right)\Xi\Phi' \;:=\; (k-1)\Xi\Phi'$$

or

$$\Xi = C\Phi^{k-1} \tag{44}$$

where C is an arbitrary elliptic function.

Now Ξ is a Darboux polynomial for X; indeed the previous relations for $X(H),\ X(\Omega),\ X(H'), X(\Omega')$ gives us the relation

$$X(\Xi) = (5n(a_1 - u) + 12u)\Xi$$

and this together with the fact that Φ is Darboux leads to the following relation involving C

$$C' = \frac{\partial}{\partial z}C = 12a_1C. \tag{45}$$

But the primitive of a non vanishing elliptic function for a generic lattice is not elliptic. We refer to [25] where the various integrals of $\wp^n, n \in \mathbb{Z}$ are computed and related to equivariant functions for modular subgroups $\Gamma \subset PSL(2,\mathbb{Z})$. More precisely $h : \mathfrak{H} \to \mathbf{P}^1$ meromorphic is equivariant under the action of Γ if

$$h\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{ah + b}{ch + d},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \gamma.$$

This forces $a_1 \equiv 0$. The conclusion is with a new constant β

$$\Omega^2 - \alpha H^3 = \beta\Phi^k$$

that is to say

$$\frac{\Phi^k}{H^3} \;=\; \frac{\Gamma - \alpha}{\beta}$$

which is another form of the equation (39). The consequence is that k is an integer satisfying $k = 6 - \frac{12}{n}$ or $n(6-k) = 12$. This is because $\Xi \in \mathbb{C}(\wp(z), \wp'(z))[u]$ and Φ is irreducible so any of its fractional powers is not a polynomial. We will examine separately the cases $n = 2$ and $n = 3$.

The main result concerning the minimal polynomial Φ are given by the following theorem

Theorem 26. *For Φ to be a minimal polynomial* (34)*, it is necessary that*

$$a_1 = 0, \quad X(\Phi) = -nu\Phi, \quad \tau_4(\Phi) = 0.$$

To give an example, we look at the case of $n = 4$. In this case

$$a_2 = -\frac{1}{3}q, \quad a_3 = \frac{1}{6}q', \quad a_4 = -\frac{1}{6}q'' + q^2,$$

so that the potential q solves an equation similar to (46): $q'' = 8q^2$.

$$u'' + 2cu^2 = 0. \tag{46}$$

It has for $c \neq 0$, the solution

$$u(z) = -\frac{3}{c}\wp_0(z)$$

where $\wp_0(z)$ is the equianharmonic Weierstrass function

$$z = \int_{\infty}^{\wp_0} \frac{dx}{2\sqrt{x^3 - 1}}.$$

For general $n = 4, 6, 12$, the potential solves the equation $q'' = 6aq^2$, with $a = \dfrac{(n-2)^2}{n-1}$.

From lemma 24 we know that a Darboux polynomial for X has necessarily all its roots satisfying the associated Riccati equation. As the degree of H is $2(n-2)$, we have to examine the cases $n = 2$ and $n = 3$ separately as in these two cases, the degree of $H(u)$ is presumably less than n.

- Take the case $n = 2$. Then

$$\Phi(u) = u^2 + 2a_1u + a_2.$$

After computation one gets $H(u) = 4a_2$ using the condition $a_1 = 0$. But $H(u)$ is a Darboux polynomial for X, so (37) $a_2' = \dfrac{\partial}{\partial z}a_2 = 0$. This forces a_2 to be an element of $\mathbb{C}$. As $a_1 = 0$, $a_2 \in \mathbb{C}$, we get a contradiction as $\Phi(u)$ is assumed to be irreducible and $\mathbb{C}$ is algebraically closed. Therefore the case $n = 2$ for $\Phi(u)$ monic irreducible is impossible.
- Next consider the case $n = 3$. Then $\Phi(u)$ assumes the form:

$$\Phi(u) = u^3 + 3a_1u^2 + 3a_2u + a_3.$$

The expression for $H(u)$ is: $18(a_2u^2 + a_3u - a_2^2)$. The previously analyzed case $n = 2$ shows that $H(u) = 18(a_2u^2 + a_3u - a_2^2)$ if $a_2 \neq 0$ necessarily factors on $\mathbb{C}(\wp(z), \wp'(z))$. Moreover a_2 can not be zero, because otherwise $H(u)$ becomes a polynomial of degree 0 or 1. Since we know that $\Phi(u)$ is irreducible, we get that 0 is a solution of our Riccati equation (a_3 can not be 0 as $\Phi(u)$ is assumed to be irreducible) which therefore takes the form $\dfrac{du}{dz} + u^2 = 0$; the general solution of the latter equation is given by $u(z) = \dfrac{1}{z + C_0}$, $C_0 \in \mathbb{C}$. We claim that such an u is not algebraic over $\mathbb{C}(\wp(z), \wp'(z))$. Indeed if that were the case, then one would get a relation of the form

$$\frac{1}{(z + C_0)^3} + a_3 = 0$$

with $a_3 \neq 0$ doubly periodic (minimal polynomial of degree 3). This is clearly absurd. Thus a_2 never vanishes.

For the case $n = 3$, the identity $\tau_4(\Phi) = 0$ disappears and only the differential system (36), with $a_1 = 0$ remains

$$\begin{aligned} 2a_2 &= -q \\ (n-2)a_3 &= -a_2' \\ 0 &= -3a_2q \end{aligned} \tag{47}$$

which gives $q'' = 3q^2$ which also of the above form with $a = \dfrac{(3-2)^2}{(3-1)} = \dfrac{1}{2}$.

Remark 27. The appearance of the potential q, $q'' = 6aq^2$ or its solution, the anharmonic Weirstrass $\wp$ function is one the unifying point of this study. Despite the similarity, it is very different from the Lamé potential.

5. Hypergeometric Functions and Conclusion

We consider the precedent differential equation for the potential

$$q^{''} = 6aq, \; a = \frac{(n-2)^2}{n-1} \tag{48}$$

with the solution $q(z) = \frac{1}{a}\wp(z) = \frac{1}{a}\wp(z;\, g_2 = 0,\, g_3)$ and we would like to make a connection with some known facts on hypergeometric functions. For commodity reason we set $g_3 = 4\lambda^6$ and the Riccati equation for u becomes

$$\frac{du}{dz} + u^2 = q(z) = \frac{1}{a}\wp(z). \tag{49}$$

By making the change of variable $X = \wp(z)$ we obtain

$$\left(\frac{dX}{dz}\right)^2 = 4X^3 - g_3 = 4(X^3 - \lambda^6) \tag{50}$$

and if we set

$$X = \lambda^2\xi, \;\; \lambda z = t, \;\; u = \lambda w \tag{51}$$

the equation (49) becomes

$$\frac{dw}{dt} + w^2 = \frac{\xi}{a}, \;\; \xi = \xi(t) \tag{52}$$

and the equation (50) transforms into

$$\left(\frac{d\xi}{dt}\right)^2 = 4(\xi^3 - 1), \quad 2dt = \frac{d\xi}{\sqrt{\xi^3 - 1}}. \tag{53}$$

The equation (52) is much more simpler than (49).
Now in the equation (52) we make the change of function

$$t = \theta(\xi), \;\; v = w\theta^{'} - \frac{\theta^{''}}{2\theta'} \tag{54}$$

which transforms a Riccati equation

$$\frac{dw}{dt} + w^2 = R(t) \tag{55}$$

into another Riccati equation

$$\frac{dv}{dt} + v^2 = r(\xi) = \theta^{'2}R(\theta) - \frac{1}{2}\{\theta, x\}, \theta^{'} = \frac{dt}{d\xi} \tag{56}$$

so that (56) is similar to a Darboux-Backlünd transformation for Riccati equations. We thus obtain a simpler form for the equation (49) and (52)

$$\frac{dv}{d\xi} + v^2 = \frac{\xi}{16}\frac{c_0\xi^3 - 4c_1}{(\xi^3 - 1)^2}, \, c_0 = \frac{4}{a} - 3, \, c_1 = \frac{1}{a} + 6. \tag{57}$$

One can show that this last equation reduces to the desired form

$$\frac{dW}{dt} + W^2 = Q(s) = \frac{1}{4}\left(\frac{\lambda^2 - 1}{s^2} + \frac{\nu^2 - 1}{(1 - s)^2} + \frac{\lambda^2 - \mu^2 + \nu^2 - 1}{s(1 - s)}\right) \tag{58}$$

with

$$\lambda = \frac{1}{3}, \;\; \mu = \frac{n}{6(n-2)}, \;\; \nu = \frac{1}{2}.$$

Before stating the conclusion, we would like to insist on the major role of the hypergeometric function in this study. We use classical notations and recall essential facts of the theory of Fuchs and Schwarz. The most general form of the hypergeometric equation is

$$y^{''} + \frac{(2 - \lambda - \mu)x + \lambda - 1}{x(1 - x)}y' + \frac{(1 - \lambda - \mu)^2 - \nu^2}{4x(1 - x)}y = 0. \tag{59}$$

We introduce the new constants a, b, c connected to the local exponents and their inverses by

$$
\begin{aligned}
&a+b+c = 1-\lambda-\mu, \quad a-b=\nu, \quad c = 1-\lambda,\\
&\mu_0 = 1-c, \quad \mu_1 = c-a-b, \quad \mu_\infty = b-a\\
&k_0 = \frac{1}{\mu_0}, \quad k_1 = \frac{1}{\mu_1}, \quad k_\infty = \frac{1}{\mu_\infty}.
\end{aligned}
$$

The associated Schwarzian equation is

$$
-4\{s,x\} = \frac{1-\mu_0^2}{x^2(1-x)} + \frac{1-\mu_1^2}{x(1-x)^2} - \frac{1-\mu_\infty^2}{x(1-x)}. \tag{60}
$$

If (k_0, k_1, k_∞) is one of the triplets $(2,2,m)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$, the monodromy group of the hypergeometric equation is of finite order $N = 2m$ (dihedral), $N = 12$ (tetrahedral), $N = 24$ (octahedral), $N = 60$ (icosahedral), with

$$
\frac{2}{N} = \frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} - 1.
$$

The main conclusion is that the cases for which the considered Riccati equation has algebraic solutions correspond exactly to those cases for which the hypergeometric equations has algebraic solutions. These correspond to platonic solids or regular polyhedra.

This is a general fact, indeed Baldassarri, Dwork and also Maier, following Klein, have shown [6, 3, 4, 5, 19] that any algebraic solution of a linear second order differential equation with regular singularities on an algebraic curve, is the pull-back via a rational map of a particular set of hypergeometric differential equations, called the basic Schwarz list. We refer to the mentioned for more details.

We finish by a remark but first we recall well known facts on modular curves : the modular function $J : \mathfrak{H} \mapsto \mathbb{C}$ gives a quotient map with respect to the projective group $PSL(2,\mathbb{R})$. It ramifies above the points $0, 1$ with ramification indices 3 and 2 respectively. The group $\Gamma(m) = \{M \in SL(2,\mathbb{Z}),\ M = \mathrm{Id} \mod m\}$ is a normal subgroup of $SL(2,\mathbb{Z})$ and has no elliptic elements for $m \geq 2$. If $Y(m)$ denotes the quotient of $\mathfrak{H}$ by $\Gamma(m)$, the cover $\mathfrak{H} \mapsto Y(m)$ is unramified and the modular function J factors over $Y(m)$, $J : Y(m) \mapsto \mathbb{C}$. The modular curve $X(m)$ is the completion of $Y(m)$ by adding to $Y(m)$ cusps so that $J : X(m) \mapsto \mathbb{P}^1$. This map ramifies above ∞ with order m. Hence the ramifications indices above $0, 1, \infty$ are $3, 2, m$. The covering group is $PSL(2, \mathbb{Z}/m\mathbb{Z})$. When $m = 3, 4, 5$ we recover the tetrahedral, octahedral and icosahedral coverings.

In the previous theorem, the vanishing of the fourth transvectant appeared as one of the conditions for a solution of a Riccati equation to be algebraic. In the analysis of this vanishing, the next theorem is of great importance and it is due to Brioschi and Wedekind.

Theorem 28 ([12, 7])**.** *Let K be an algebraically closed field of characteristic zero. The fourth transvectant $(f,f)^{(4)}$ of a non-trivial binary form f of order $k \geq 4$ is identically zero if and only if f is $GL(2,K)$-equivalent to one of the following forms*

(1) x_1^k *or* $x_1^{k-1}x_2$ (degenerate case)
(2) $x_2(x_1^3 + x_2^3)$ (tetrahedral case)
(3) $x_1x_2\left(x_1^4 + x_2^4\right)$ (octahedral case)
(4) $x_1x_2\left(x_1^{10} - 11x_1^5x_2^5 - x_2^{10}\right)$ (icosahedral case).

The main point here is that the vanishing of the fourth transvectant of a non-degenerate binary form f forces it to be one of the Klein forms.

6. Note

There are various way one could possibly extend the study realized in this paper. First we may consider the case of homogeneous linear third order or n-th order

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0,$$

$n > 2$ over $(\mathcal{C}(x), \delta)$, $\delta(x) = 1$ and $\mathcal{C}$ algebraically closed. Using the substitution $u = \dfrac{y'}{y}$ one reduces the homogeneous linear differential equation to a non-linear differential equation of order $n-1$. Do there exist anharmonics in this case? Moreover does the Darboux formalism in the last section of this paper generalize to the case of Riccati equations on the field of functions of an algebraic curve over $\mathbb{C}$ (the case we studied is essentially the case of elliptic curves). Finally one possible further generalization of the Darboux formalism may occur through the study of the general Abel differential equations

$$\frac{du}{dz} = u^3 + q$$

with q in some function field.

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